TOPOLOGICAL ENTROPY OF SOME AUTOMORPHISMS OF REDUCED AMALGAMATED FREE PRODUCT C*–ALGEBRAS

Kenneth J. Dykema

20 May 1999

Abstract. Certain classes of automorphisms of reduced amalgamated free products of C*–algebras are shown to have Brown–Voiculescu topological entropy zero. Also, for automorphisms of exact C*–algebras, the Connes–Narnhofer–Thirring entropy is shown to be bounded above by the Brown–Voiculescu entropy. These facts are applied to generalize Størmer's result about the entropy of automorphisms of the II_1–factor of a free group.

§1. Introduction.

Kolmogorov's entropy invariant was extended by Connes and Sørmer [**5**] to an invariant $h_\tau(\alpha)$ for an automorphism α of a von Neumann algebra with a given normal faithful tracial state τ which is preserved by the automorphism. One of the several results about the Connes–Størmer entropy (see [**12**] for a survey) is Størmer's result [**11**] that the free shift on $L(F_\infty)$ has entropy zero. Here $L(F_\infty)$ is the II_1–factor defined by the left regular representation of the free group F_∞ on countably infinitely many generators. More generally, Størmer's theorem states that the entropy of σ_* is zero whenever σ_* is the automorphism of $L(F_\infty)$ induced by a permutation σ of the generators of F_∞ that has neither fixed points nor finite cycles; the free shift is the automorphism σ_* where, when the generators of F_∞ are indexed by the integers, σ corresponds to the shift $n \mapsto n+1$.

The Connes–Størmer entropy was extended by Connes, Narnhofer and Thirring [**4**] to an invariant, generally referred to as the CNT–entropy and denoted $h_\phi(\alpha)$, for an automorphism α of a unital C*–algebra A with respect to an α–invariant state ϕ of A. See also Sauvageot's and Thouvenot's modification [**10**], giving an entropy that is in general bounded above by the CNT–entropy and that coincides with the CNT–entropy when the C*–algebra A is nuclear. Theorem VII.2 of [**4**] shows that given an automorphism α of a C*–algebra A preserving a state ϕ, if $\mathcal{M}$ is the von Neumann algebra generated by the image of A under the GNS representation

1991 *Mathematics Subject Classification.* 46L55.

of ϕ, if $\overline{\alpha}$ and $\overline{\phi}$ are the canonical extensions of α and ϕ to $\mathcal{M}$, then $h_{\overline{\phi}}(\overline{\alpha}) = h_\phi(\alpha)$. (Their theorem is stated only for nuclear A and hyperfinite $\mathcal{M}$, but their proof applies generally.)

A noncommutative topological entropy was invented by Voiculescu [**15**] for automorphisms of nuclear C*–algebras; N. Brown [**2**] extended it to handle automorphisms of exact C*–algebras. This Brown–Voiculescu entropy of an automorphism α is denoted $ht(\alpha)$. Voiculescu proved that if α is an automorphism of a unital nuclear C*–algebra A and if ϕ is an α–invariant state then $h_\phi(\alpha) \le ht(\alpha)$. Here we show (Proposition 9) that the same inequality holds when A is a unital exact C*–algebra.

In [**7**], we proved that every reduced amalgamated free product of exact C*–algebras gives an exact C*–algebra. In this note, we build upon that proof to show that certain classes of automorphisms of C*–algebras arising as reduced amalgamated free products have zero topological entropy.

The following section is the main part of the paper and contains the results and their proofs. At the end of it are two questions.

I would like to thank the members of the Institute of Mathematics in Luminy, France and of the Erwin Schrödinger Insitute in Vienna, where much of this research was done, for their hospitality; I would like to thank especially J.B. Cooper for organizing the Schrödinger Institute's concentration in Fucntional Analysis in 1999. Finally, the financial support of the CNRS of France and of the Schrödinger Institute is gratefully acknowledged.

§2. Entropy of Automorphisms.

Theorem 1. *Let B be a finite dimensional C*–algebra, let I be a set and for every $\iota \in I$ let A_ι be a finite dimensional C*–algebra containing B as a unital C*–subalgebra and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota)$$

be the reduced amalgamated free product of C*–algebras and denote the embeddings arising from the free product construction by $\lambda_\iota : A_\iota \hookrightarrow A$. Let σ be a permutation of I such that for every $\iota \in I$ there is a $*$–isomorphism $\alpha_\iota : A_\iota \to A_{\sigma(\iota)}$ such that $\alpha_\iota(B) = B$ and $\phi_{\sigma(\iota)} \circ \alpha_\iota = \alpha_\iota \circ \phi_\iota$. Assume further that the automorphism $\alpha_\iota \restriction_B$ of B is independent of $\iota \in I$, and call this automorphism β. There is a unique automorphism α of A such that $\alpha \circ \lambda_\iota = \lambda_{\sigma(\iota)} \circ \alpha_\iota$ for all $\iota \in I$.

Then $ht(\alpha) = 0$.

Proof. In Voiculescu's construction [**14**] of the reduced amalgamated free product C*–algebra A, one takes the Hilbert B–module $E_\iota = L^2(A_\iota,\phi_\iota)$ on which A_ι acts via the GNS representation, one lets $\xi_\iota = \widehat{1_{A_\iota}} \in E_\iota$, where $A_\iota \ni a \mapsto \widehat{a} \in E_\iota$ is the defining map, one lets $E_\iota^{\mathrm{o}} = E_\iota \ominus \xi_\iota B$, one constructs the free product of Hilbert B–modules $(E,\xi) = \mathop{*}_{\iota\in I}(E_\iota,\xi_\iota)$, given by

$$E = \xi B \oplus \bigoplus_{\substack{n\geq 1 \\ \iota_1,\iota_2\ldots,\iota_n\in I \\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\ldots,\iota_{n-1}\neq\iota_n}} E_{\iota_1}^{\mathrm{o}} \otimes_B E_{\iota_2}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}},$$

and one defines A acting on E; (see [**7**, §1] for Voiculescu's construction in the notation used here). The $*$–isomorphism $\alpha_\iota : A_\iota \to A_{\sigma(\iota)}$ gives rise to an invertible and isometric linear map $U_\iota : E_\iota \to E_{\sigma(\iota)}$ given by $U_\iota\widehat{a} = \widehat{\alpha_\iota(a)}$, (but note that U_ι need not be B–linear). Taking A_ι, respectively $A_{\sigma(\iota)}$, acting via its GNS representation on E_ι, respectively $E_{\sigma(\iota)}$, we have for $a \in A_\iota$ that $U_\iota a U_\iota^{-1} = \alpha_\iota(a)$. Having assumed that $\alpha_\iota\restriction_B = \beta$ is independent of ι, we see that the collection of isometries $(U_\iota)_{\iota\in I}$ gives rise to an isometric and invertible linear map $U : E \to E$ given by $U\xi b = \xi\beta(b)$ for $b \in B$ and $U(\zeta_1 \otimes \cdots \otimes \zeta_n) = (U_{\iota_1}\zeta_1) \otimes \cdots \otimes (U_{\iota_n}\zeta_n)$ for $\zeta_j \in E_{\iota_j}^{\mathrm{o}}$ with $\iota_1,\ldots,\iota_n \in I$ and $\iota_1 \neq \iota_2,\ldots,\iota_{n-1} \neq \iota_n$. The automorphism α of A is then defined by $\alpha(x) = UxU^{-1}$.

Let π denote the inclusion, arising from the free product construction, of A in $\mathfrak{L}(E)$. We will show that $ht(\pi,\alpha) = 0$, and in order to do so we must show that $ht(\pi,\alpha,\omega,\delta) = 0$ for every finite subset ω of A and every $\delta > 0$. But for this it will suffice to let ω be a finite subset of any given set whose linear span is a dense subset of A. The set W of reduced words in $(A_\iota)_{\iota\in I}$ has dense linear span in A, and we will take $\omega \subseteq W$, where a *reduced word* is (an element of A given by) an expression of the form $a_1a_2\cdots a_n$, where $n \geq 1$, $a_j \in A_{\iota_j} \cap \ker\phi_{\iota_j}$ and $\iota_1 \neq \iota_2,\ldots,\iota_{n-1} \neq \iota_n$; we call n the *length* of the reduced word and we call the set $\{\iota_1,\ldots,\iota_n\} \subseteq I$ the *alphabet* for the word; we consider elements of B to be reduced words of length 0 and with alphabet equal to the empty set. If $\omega \subseteq W$ we define the alphabet for ω to be the union of the alphabets of the elements of ω.

Let J be a subset of I and let $(A^{(J)},\phi^{(J)}) = \mathop{*}_{\iota\in J}(A_\iota,\phi_\iota)$ be the reduced amalgamated free product of the subfamily. Then $A^{(J)}$ acts canonically on the Hilbert B–module $E^{(J)}$, where $(E^{(J)},\xi) = \mathop{*}_{\iota\in J}(E_\iota,\xi_\iota)$. We will presently show in detail that $A^{(J)}$ is naturally embedded into A and that there is a conditional expectation from A onto $A^{(J)}$. Note that $E^{(J)}$ is a complemented submodule of E; let $\Theta^{(J)} : \mathfrak{L}(E) \to \mathfrak{L}(E^{(J)})$ be given by compression. Consider

the Hilbert B–module

$$E(J)=\eta B\oplus\bigoplus_{\substack{n\geq 1\\ \iota_1,\iota_2\ldots,\iota_n\in I\\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\ldots,\iota_{n-1}\neq\iota_n\\ \iota_1\notin J}} E^{\mathrm{o}}_{\iota_1}\otimes_B E^{\mathrm{o}}_{\iota_2}\otimes_B\cdots\otimes_B E^{\mathrm{o}}_{\iota_n},$$

where ηB is simply a copy of B considered as a Hilbert B–module with η denoting the identity element of B. There is then a unitary $V_J:E\to E^{(J)}\otimes_B E(J)$ given by erasing parenthesis and absorbing η, analogous to the unitary $E\to E_\iota\otimes_B E(\iota)$ in Voiculescu's construction of the reduced amalgamated free product; this unitary provides an embedding $i^{(J)}:\mathcal{L}(E^{(J)})\to\mathcal{L}(E)$ given by $i^{(J)}(x)=V_J^*(x\otimes 1)V_J$, which then satisfies that $\Theta^{(J)}\circ i^{(J)}$ is the identity on $\mathcal{L}(E^{(J)})$. Moreover, note that $i^{(J)}$ takes a reduced word considered as an element of $A^{(J)}$ to the same reduced word considered as an element of A. Hence $A^{(J)}$ is embedded into A via $i^{(J)}$, and $\Theta^{(J)}$ provides a conditional expectation from A onto the embedded copy of $A^{(J)}$.

Let $\omega\subseteq W$ be a finite set of reduced words and let $\delta>0$; we will find an upper bound for $rcp(\pi,\omega,\delta)$. Let q be the maximum of the lengths of the words belonging to ω and let J be the alphabet for ω, which is thus a finite set. Given $k\in\mathbf{N}$, consider the complemented submodule of $E^{(J)}$,

$$E^{(J)}_{(\to k)}=\xi B\oplus\bigoplus_{\substack{1\leq n\leq k\\ \iota_1,\iota_2\ldots,\iota_n\in J\\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\ldots,\iota_{n-1}\neq\iota_n}} E^{\mathrm{o}}_{\iota_1}\otimes_B E^{\mathrm{o}}_{\iota_2}\otimes_B\cdots\otimes_B E^{\mathrm{o}}_{\iota_n}$$

and let $\Phi^{(J)}_k:\mathcal{L}(E^{(J)})\to\mathcal{L}(E^{(J)}_{(\to k)})$ be given by compression. In [**7**, 3.1], unital completely positive maps $\Psi^{(J)}_k:\mathcal{L}(E^{(J)}_{(\to k)})\to\mathcal{L}(E^{(J)})$ were constructed so that for every $a\in A^{(J)}$, $\lim_{k\to\infty}\|a-\Psi^{(J)}_k\circ\Phi^{(J)}_k(a)\|=0$. Furthermore, from the proof of [**7**, 3.1] we see that for every $\epsilon>0$ and every $q\in\mathbf{N}$ there is $k_0(\epsilon,q)\in\mathbf{N}$ such that for every reduced word $a\in A^{(J)}$ of length no greater than q, if $k\geq k_0(\epsilon,q)$ then $\|a-\Psi^{(J)}_k\circ\Phi^{(J)}_k(a)\|\leq\epsilon\|a\|$; moreover, $k_0(\epsilon,q)$ is universal, in the sense that it is the same for all J and all families $\big((A_\iota,\phi_\iota)\big)_{\iota\in J}$. Hence, under the same conditions, $\|a-i^{(J)}\circ\Psi^{(J)}_k\circ\Phi^{(J)}_k\circ\Theta^{(J)}(a)\|\leq\epsilon\|a\|$. Let us write $\widetilde{\Phi}^{(J)}_k$ for the composition $\Phi^{(J)}_k\circ\Theta^{(J)}:\mathcal{L}(E)\to\mathcal{L}(E^{(J)}_{(\to k)})$ and $\widetilde{\Psi}^{(J)}_k$ for the composition $i^{(J)}\circ\Psi^{(J)}_k:\mathcal{L}(E^{(J)}_{(\to k)})\to\mathcal{L}(E)$. Let $\epsilon=\delta/\max\{\|a\|\mid a\in\omega\}$, let q be the maximum of the lengths of the words belonging to ω and let $k=k_0(\epsilon,q)$. Since J is a finite set and since each E_ι is a finite dimensional complex vector space, the Hilbert B module $E^{(J)}_{(\to k)}$ is a finite dimensional vector space; hence the C*–algebra $\mathcal{L}(E^{(J)}_{(\to k)})$ is finite dimensional. Taking the unital completely positive maps $\widetilde{\Phi}^{(J)}_k$ and $\widetilde{\Psi}^{(J)}_k$, we see that $rcp(\pi,\omega,\delta)\leq\operatorname{rank}\big(\mathcal{L}(E^{(J)}_{(\to k)})\big)$. We now perform a crude (but

sufficient) estimate of this rank. Let $d(J)$ be the maximum over $\iota \in J$ of the dimension of E_ι as a vector space; then we can estimate

$$\dim(E^{(J)}_{(\to k)}) \leq \dim(B) + \sum_{n=1}^{k} |J|^n d(J)^n \leq \dim(B) + k|J|^k d(J)^k.$$

Let ρ be a faithful representation of B on a finite dimensional Hilbert space $\mathcal{V}$. Then the C*–algebra $\mathcal{L}(E^{(J)}_{(\to k)})$ is faithfully represented on the Hilbert space $E^{(J)}_{(\to k)} \otimes_\rho \mathcal{V}$, which has dimension $\leq \dim(E^{(J)}_{(\to k)}) \dim(\mathcal{V})$. Thus we have

$$rcp(\pi, \omega, \delta) \leq \big(\dim(B) + k|J|^k d(J)^k\big) \dim(\mathcal{V}).$$

Now we are in a position to show that

$$ht(\pi, \alpha, \omega, \delta) = 0. \tag{1}$$

Given the nature of our automorphism α, for every $n \in \mathbf{N}$ the maximum length and the maximum norm of words belonging to

$$\omega \cup \alpha(\omega) \cup \cdots \cup \alpha^{n-1}(\omega) \tag{2}$$

are the same as for ω, and we may choose $k = k_0(q, \epsilon)$ as for ω above. However, the alphabet J_n of the set of words (2) is equal to $J \cup \sigma(J) \cup \cdots \cup \sigma^{n-1}(J)$, and thus $|J_n| \leq n|J|$. But the existence of the isomorphisms α_ι preserving conditional expectations implies that $\dim(E_{\sigma(\iota)}) = \dim(E_\iota)$, and hence $d(J_n) = d(J)$. Hence we have the estimate

$$rcp(\pi, \omega \cup \alpha(\omega) \cup \cdots \cup \alpha^{n-1}(\omega), \delta) \leq \big(\dim(B) + kn^k |J|^k d(J)^k\big) \dim(\mathcal{V}).$$

As the upper bound grows subexponentially in n, the estimate implies (1).

□

We now list as corollaries some particular sorts of automorphisms to which the above theorem applies. First we have free products of automorphisms, which correspond to when the permutation σ in Theorem 1 is the identity.

Corollary 2. *Let*

$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota)$$

be the reduced amalgamated free product of finite dimensional C*–algebras as in the statement of Theorem 1. For every $\iota \in I$ let $\alpha_\iota \in \mathrm{Aut}(A_\iota)$ be such that $\alpha_\iota(B) = B$, $\phi_\iota \circ \alpha_\iota = \alpha_\iota \circ \phi_\iota$;

suppose that the automorphism $\alpha_\iota\restriction_B$ of B is the same for all $\iota \in I$. Let $\alpha = \mathop{*}\limits_{\iota\in I} \alpha_\iota \in \mathrm{Aut}(A)$; by this we mean that α is the automorphism of A that when restricted to the naturally embedded copy of A_ι in A is α_ι.

Then $ht(\alpha) = 0$.

Next we have the free shifts and their analogues for general permutations.

Definition 3. If $(A,\phi) = \mathop{*}\limits_{\iota\in I} (A_\iota,\phi_\iota)$ is a reduced amalgamated free product of C*–algebras, where each (A_ι,ϕ_ι) is a copy of a fixed pair (D,ψ) of a unital exact C*–algebra D and a conditional expectation ψ from D onto a unital C*–subalgebra B having faithful GNS representation, and if σ is a permutation of the index set I, then what we call the corresponding *free permutation* is the automorphism σ_* of A sending the embedded copy of A_ι in A identically to the embedded copy of $A_{\sigma(\iota)}$ in A, for every $\iota \in I$.

We say that the pair (D,ψ) has the *ZEFP property* (with respect to ht) if $ht(\sigma_*) = 0$ whenever σ_* is a free permutation of a free product of some copies of (D,ψ).

The acronym ZEFP is for “zero entropy free permutation.”

Corollary 4. *Let B and D be finite dimensional C*–algebras with B contained as a unital C*–subalgebra of D; let $\psi : D \to B$ be a conditional expectation whose GNS representation is faithful. Then (D,ψ) has the ZEFP property.*

Corollary 5. *Let J be a set, let B be a finite dimensional C*–algebra and for every $\iota \in J$ let D_ι be a finite dimensional C*–algebra and $\psi_\iota : D_\iota \to B$ is a conditional expectation having faithful GNS representation. Let $(D,\psi) = \mathop{*}\limits_{\iota\in J} (D_\iota,\psi_\iota)$. Then (D,ψ) has the ZEFP property*

Proof. If I is a set and if σ is a permutation of I, let σ_* be the corresponding free permutation of the free product of $|I|$ copies of (D,ψ). Then σ_* is in the obvious way equal to a free permutation of a reduced free product of finite dimensional C*–algebras, corresponding to the permutation $\sigma \times \mathrm{id}_J$ of $I \times J$. Thus $ht(\sigma_*) = 0$ by Theorem 1.

Definition and Proposition 6. *Let (D,ψ) and $(\widetilde{D},\tilde{\psi})$ be pairs of a unital exact C*–algebras D and $\widetilde{D}$ with conditional expectations ψ from D onto a unital C*–subalgebra $B \subseteq D$ and $\tilde{\psi}$ from $\widetilde{D}$ onto a unital C*–subalgebra $\widetilde{B} \subseteq \widetilde{D}$, whose GNS representations are faithful. We say (D,ψ) is* included *in $(\widetilde{D},\tilde{\psi})$, and write $(D,\psi) \subseteq (\widetilde{D},\tilde{\psi})$, if D is a C*–subalgebra of $\widetilde{D}$ in such a way that $B \subseteq \widetilde{B}$ and $\tilde{\psi}\restriction_D = \psi$. We call the inclusion $(D,\psi) \subseteq (\widetilde{D},\tilde{\psi})$* unital *if D is a unital C*–subalgebra of $\widetilde{D}$.*

If $(D,\psi)\subseteq(\widetilde{D},\tilde{\psi})$ and if $(\widetilde{D},\tilde{\psi})$ has the ZEFP property then (D,ψ) has the ZEFP property.

Proof. First suppose that the inclusion is unital. By the main result of [**1**], the free product of $|I|$ copies of (D,ψ) embeds in the free product of $|I|$ copies of $(\widetilde{D},\tilde{\psi})$. Let σ be a permutation of I, let σ_* be corresponding free permutation of the free product of $|I|$ copies of (D,ψ) and let $\widetilde{\sigma}_*$ be the free permutation of the free product of $|I|$ copies of $(\widetilde{D},\tilde{\psi})$. Then σ_* is the restriction of $\widetilde{\sigma}_*$. As the Brown–Voiculescu topological entropy is monotone [**2**, 2.1], we have $ht(\sigma_*)=0$; hence (D,ψ) has the ZEFP property.

If the inclusion $(D,\psi)\subseteq(\widetilde{D},\tilde{\psi})$ is nonunital, let $p\in\widetilde{D}$ denote the identity element of D and let 1 denote the identity element of $\widetilde{D}$; then $1-p\in\widetilde{B}$. Let $D'=D+\mathbf{C}(1-p)\subseteq\widetilde{D}$ and let $B'=B+\mathbf{C}(1-p)\subseteq\widetilde{B}$; then for $d\in D$ and $\lambda\in\mathbf{C}$, $\tilde{\psi}(d+\lambda(1-p))=\psi(d)+\lambda(1-p)$; let $\psi'=\tilde{\psi}\restriction_{D'}:D'\to B'$. Then by the unital case just proved, (D',ψ') has the ZEFP property. Let I be a set and let $(A',\phi')=\underset{\iota\in I}{*}(A'_\iota,\phi'_\iota)$ where each (A'_ι,ϕ'_ι) is a copy of (D',ψ'); let $(A,\phi)=\underset{\iota\in I}{*}(A_\iota,\phi_\iota)$ where each (A_ι,ϕ_ι) is a copy of (D,ψ). Then $p\in B'\in A'$ and A is canonically isomorphic to $pA'p$; if σ_* is a free permutation on A corresponding to a permutation σ of I, then σ_* is the restriction of the corresponding free permutation σ'_* of A' to $pA'p$. Again by monotonicity, we see that $ht(\sigma_*)=0$ and (D,ψ) has the ZEFP property. □

Application of Corollary 5 and Proposition 6 leads to many examples, a few of which are below.

Examples 7. *The following pairs have the ZEFP property.*

(i) $(\mathcal{T},\phi_1)$ *where $\mathcal{T}$ is the Toeplitz algebra, which is generated by a nonunitary isometry v, and where ϕ_1 is the state on $\mathcal{T}$ satisfying $\phi_1(vv^*)=0$;*

(ii) $(\mathcal{O}_\infty,\phi)$ *where $\mathcal{O}_\infty$ is the Cuntz algebra [**6**], which is generated by isometries $s_1,s_2,\ldots$ having orthogonal ranges, and where ϕ is the state on $\mathcal{O}_\infty$ such that $\phi(s_js_j^*)=0$ for all j;*

(iii) $(\mathcal{O}_n,\phi_n)$, *with $n\in\mathbf{N}$, $n\geq 2$, where $\mathcal{O}_n$ is the Cuntz algebra [**6**], which is generated by isometries $s_1,\ldots,s_n$, whose range projections sum to 1, and where, for any fixed choice of $\gamma_1,\ldots,\gamma_n\in[0,1]$ such that $\gamma_1+\cdots\gamma_n=1$, ϕ_n is the state on $\mathcal{O}_n$ given by*

$$\phi_n(s_{i_1}s_{i_2}\cdots s_{i_k}s_{j_\ell}^*\cdots s_{j_2}^*s_{j_1}^*)=\begin{cases}\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k} & \text{if } k=\ell,\ i_1=j_1,\ldots,i_k=j_k\\ 0 & \text{otherwise;}\end{cases}\tag{3}$$

(iv) $(\mathcal{O}_\infty,\phi_\infty)$ *where $\mathcal{O}_\infty$ is generated by isometries $s_1,s_2,\ldots$ having orthogonal ranges*

and where for any fixed choice of $\gamma_1, \gamma_2, \ldots \in [0,1]$ *such that* $\sum_1^\infty \gamma_j \leq 1$*,* ϕ_∞ *is the state on* $\mathcal{O}_\infty$ *satisfying (3);*

(v) $(C(\mathbf{T}), \tau)$ *where* $\mathbf{T}$ *is the circle and where the state* τ *is given by Lebesgue measure on* $\mathbf{T}$.

Proof. For (i), let $D_1 = \mathbf{C} \oplus \mathbf{C}$ with minimal projection $p \in D_1$ and let ψ_1 be the state on D_1 such that $\psi_1(p) = 1/2$; let $D_2 = M_2(\mathbf{C})$ with a system of matrix units $(e_{ij})_{1 \leq i,j \leq 2}$ in D_2 and let ψ_2 be the state on D_2 so that $\psi_2(e_{11}) = 1$. Let $(\widetilde{D}, \tilde{\psi}) = (D_1, \psi_1) * (D_2, \psi_2)$. Considering the unitary $u = 1 - 2p \in D_1$, we see that $L^2(D_1, \psi_1)$ has orthonormal basis $\{\widehat{1_{D_1}}, \hat{u}\}$; moreover, $L^2(D_2, \psi_2)$ has orthonormal basis $\{\widehat{1_{D_2}}, \hat{e}_{21}\}$. Therefore, $L^2(\widetilde{D}, \tilde{\psi})$ has orthonormal basis

$$\{\xi\} \cup \{\hat{u},\, \hat{u} \otimes \hat{e}_{21},\, \hat{u} \otimes \hat{e}_{21} \otimes \hat{u},\, \hat{u} \otimes \hat{e}_{21} \otimes \hat{u} \otimes \hat{e}_{21},\, \ldots\} \quad \cup$$
$$\cup \quad \{\hat{e}_{21},\, \hat{e}_{21} \otimes \hat{u},\, \hat{e}_{21} \otimes \hat{u} \otimes \hat{e}_{21},\, \hat{e}_{21} \otimes \hat{u} \otimes \hat{e}_{21} \otimes \hat{u},\, \ldots\},$$

where $\xi = \widehat{1_{\widetilde{D}}}$; moreover, $\tilde{\psi}$ is the vector state associated to ξ. Let $v = e_{21} u e_{22} + e_{11} u e_{21} \in \widetilde{D}$. Then v is an isometry satisfying

$$\begin{aligned} v : \xi &\mapsto \hat{u} \otimes \hat{e}_{21} \\ \hat{u} \otimes (\cdots) &\mapsto \hat{u} \otimes \hat{e}_{21} \otimes \hat{u} \otimes (\cdots) \\ \hat{e}_{21} \otimes (\cdots) &\mapsto \hat{e}_{21} \otimes \hat{u} \otimes \hat{e}_{21} \otimes (\cdots). \end{aligned}$$

Thus the C*–subalgebra of $\widetilde{D}$ generated by v is isomorphic to $\mathcal{T}$ and, as ξ is orthogonal to the range space of v, the restriction of $\tilde{\psi}$ to the copy of $\mathcal{T}$ is the state ϕ_1 described in (i). Now Corollary 5 and Proposition 6 imply that $(\mathcal{T}, \phi_1)$ has the ZEFP property.

Note that (ii) is a special case of (iv). However, for future reference we would like to point out how (ii) follows from (i). From [**14**, §2] (or see [**16**, 1.5.10]), $(\mathcal{O}_\infty, \phi)$ is the free product of countably infinitely many copies of $(\mathcal{T}, \phi_1)$. Hence by Corollary 5 and Proposition 6, $(\mathcal{O}_\infty, \phi)$ has the ZEFP property.

For (iii), let $\widetilde{B} = \mathbf{C} \oplus \mathbf{C}$ with minimal projection p; let $D_1 = M_2(\mathbf{C})$ with a system of matrix units $(e_{ij})_{0 \leq i,j \leq 1}$, with $\widetilde{B}$ unitally embedded by identifying p and e_{11}, and with conditional expectation $\psi_1 : D_1 \to \widetilde{B}$ given by

$$\psi_1\Big(\sum_{i,j=0}^{1} c_{ij} e_{ij}\Big) = c_{11} p + c_{00}(1-p);$$

let $D_2 = M_{n+1}(\mathbf{C})$ with a system of matrix units $(f_{ij})_{0 \leq i,j \leq n}$, with $\widetilde{B}$ unitally embedded by identifying $1-p$ and f_{00} and with conditional expectation $\psi_2 : D_2 \to \widetilde{B}$ given by

$$\psi_2\Big(\sum_{i,j=0}^{n} c_{ij} f_{ij}\Big) = \Big(\sum_{j=1}^{n} \gamma_j c_{jj}\Big) p + c_{00}(1-p).$$

Let

$$(\widetilde{D},\tilde{\psi})=(D_1,\psi_1)*(D_2,\psi_2).$$

For every $k\in\{1,\ldots,n\}$, let $s_k=f_{k0}e_{01}\in\widetilde{D}$. Then $s_k^*s_k=p$ and $s_ks_k^*=f_{kk}$. In $p\widetilde{D}p$, $s_1,\ldots,s_n$ are isometries with range projections summing to p, so they generate a copy of $\mathcal{O}_n$ in $\widetilde{D}$ with identity element p and to which the conditional expectation $\tilde{\psi}$ restricts to a state, ϕ_n (when $\mathbf{C}p$ is identified with $\mathbf{C}$). It is clear that $\phi_n(s_js_j^*)=\gamma_j$; in order to see that (3) holds, one can argue by induction on k and use freeness. Now Corollary 5 and Proposition 6 imply that $(\mathcal{O}_n,\phi_n)$ has the ZEFP property.

The proof of (iv) is similar to the that of (iii), but taking D_2 to be the unitization of the C*–algebra, $\mathcal{K}$, of compact operators on separable infinite dimensional Hilbert space. Letting $(f_{ij})_{i,j\geq 0}$ be a system of matrix units for $\mathcal{K}$, embed $\widetilde{B}$ in D_2 by identifying $1-p$ and f_{00}, and let $\psi_2:D_2\to\widetilde{B}$ be the conditional expectation given by

$$\psi_2(1)=1\qquad \psi_2(f_{jj})=\gamma_jp\quad(j\geq 1)\qquad \psi_2(f_{00})=1-p.$$

Then letting $s_j=f_{j0}e_{01}$, $(j\geq 1)$ we have $s_j^*s_j=p$ and $s_js_j^*=f_{jj}$; hence $\{s_1,s_2\ldots\}$ generates a copy of $\mathcal{O}_\infty$ in $p\widetilde{D}p$, to which the restriction of $\tilde{\psi}$ is seen to be ϕ_∞ as described in (iv) above.

For (v), it is only required to apply Corollary 5 and Proposition 6 after noting that the choice of an infinite order element in the group $\mathbf{Z}_2*\mathbf{Z}_2$, (the free product of the two–element group with itself), gives rise to an canonical trace preserving embedding of the reduced group C*–algebra $C_r^*(\mathbf{Z})\cong C(\mathbf{T})$ in the reduced group C*–algebra $C_r^*(\mathbf{Z}_2*\mathbf{Z}_2)$, which in turn arises as the reduced free product

$$(C_r^*(\mathbf{Z}_2*\mathbf{Z}_2),\tau_{\mathbf{Z}_2*\mathbf{Z}_2})=(C_r^*(\mathbf{Z}_2),\tau_{\mathbf{Z}_2})*(C_r^*(\mathbf{Z}_2),\tau_{\mathbf{Z}_2})$$

of finite dimensional C*–algebras.

□

Example 7(i), can be used to give another proof of Brown's and Choda's result that the free permutations on the Cuntz algebra $\mathcal{O}_\infty$ have topological entropy zero.

Proposition 8. ([**3**]) *Let $\{s_1,s_2,\ldots\}$ be a family of isometries having orthogonal ranges and generating the Cuntz algebra $\mathcal{O}_\infty$; let α be an automorphism of $\mathcal{O}_\infty$ given by $\alpha(s_k)=s_{\sigma(k)}$, where σ is some permutation of $\mathbf{N}$. Then $ht(\alpha)=0$.*

Proof. As mentioned in the proof of 7(ii) above, $\mathcal{O}_\infty$ is the free product of countably infinitely many copies of $(\mathcal{T},\phi_1)$ as in 7(i), indexed by $\mathbf{N}$. We get $ht(\alpha)=0$ because $(\mathcal{T},\phi_1)$ has the

ZEFP property.

□

We will use Example 7(v) to generalize, to the case of arbitrary permutations, Størmer's result [**11**] about free shifts on $L(F_\infty)$. For this, we need to extend Voiculescu's inequality $h_\sigma(\alpha) \leq ht(\alpha)$ to the case of automorphisms of unital exact C*–algebras. The proof below is inspired by Voiculescu's [**15**, 4.6]; we refer to [**4**] and [**2**] for relevant concepts and definitions.

Proposition 9. *Let A be a unital exact C*–algebra, let $\alpha \in \mathrm{Aut}(A)$ and let σ be a state on A satisfying $\sigma \circ \alpha = \sigma$. Then $h_\sigma(\alpha) \leq ht(\alpha)$.*

Proof. Let $\gamma : M_k(\mathbf{C}) \to A$ be a unital completely positive map. Let ω be a finite subset of A such that $\gamma(M_k(\mathbf{C})) \subseteq \mathrm{span}\omega$ and

$$\gamma\big(\{x \in M_k(\mathbf{C}) \mid \|x\| \leq 1\}\big) \subseteq \Big\{\sum_{x\in\omega} \lambda(x)x \;\Big|\; \lambda(x) \in \mathbf{C},\, \sum_{x\in\omega} |\lambda(x)| \leq 1\Big\};$$

for future reference, assume that also the identity element of A belongs to ω. Let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation of A on a Hilbert space $\mathcal{H}$. Let $\delta > 0$ and $n \in \mathbf{N}$ and suppose that D is a finite dimensional C*–algebra and that $\phi : A \to D$ and $\psi : D \to \mathcal{L}(\mathcal{H})$ are unital completely positive maps such that $\forall a \in \omega \cup \alpha(\omega) \cup \cdots \cup \alpha^{n-1}(\omega)$, $\|\psi \circ \phi(a) - \pi(a)\| < \delta$. Then for all $x \in M_k(\mathbf{C})$ with $\|x\| \leq 1$ and for all $j \in \{0, 1, \ldots, n-1\}$,

$$\|\psi \circ \phi \circ \alpha^j \circ \gamma(x) - \pi \circ \alpha^j \circ \gamma(x)\| < \delta.$$

Let C be the C*–algebra generated by $\pi(A) \cup \psi(D)$. Consider an abelian model, call it $\mathfrak{A}$, for $\big(A, \phi, (\alpha^j \circ \gamma)_{j=0}^{n-1}\big)$ consisting of an abelian finite dimensional C*–algebra B, a unital completely positive map $P : A \to B$, a state μ on B such that $\mu \circ P = \sigma$ and $*$–subalgebras $B_1, \ldots, B_n$ of B. There is a unital completely positive map $P' : C \to B$ such that $P' \circ \pi = P$. If $E_j : B \to B_j$ are the canonical conditional expectations with respect to μ, then letting

$$\begin{aligned} \rho_j &= E_j \circ P \circ \alpha^j \circ \gamma : A \to B_j \\ \rho_j' &= E_j \circ P' \circ \psi \circ \phi \circ \alpha^j \circ \gamma : A \to B_j, \end{aligned}$$

we have $\|\rho_j - \rho_j'\| \leq \delta$ for all j. Then by [**4**, IV.2], $|s_\mu(\rho_j) - s_\mu(\rho_j')| < \eta$ where $\eta = 3\delta + 6\delta\log(1 + k^2\delta^{-1})$. Let $\sigma' = \mu \circ P'$ and let $\mathfrak{A}'$ be the abelian model for $\big(C, \sigma', (\psi \circ \phi \circ \alpha^j \circ \gamma)_{j=0}^{n-1}\big)$ consisting of $(B, \mu, B_1, \ldots, B_n)$ and the completely positive map $P' : C \to B$. Then from equation (III.3) of [**4**], the entropy of the abelian model $\mathfrak{A}$ differs from that of $\mathfrak{A}'$ by no more than $n\eta$. Moreover,

the entropy of the abelian model $\mathfrak{A}'$ is bounded above by $H_{\sigma'}\big((\psi\circ\phi\circ\alpha^j\circ\gamma)_{j=0}^{n-1}\big)$; this is by [**4**, III.6(a,c)] bounded above by $H_{\sigma'}(\psi)$, which is $\leq \log \mathrm{rank}(D)$. We may choose (D,ϕ,ψ) so that $\mathrm{rank}(D) \leq rcp(\pi, \omega\cup\alpha(\omega)\cup\cdots\cup\alpha^{n-1}(\omega), 4\delta)$; indeed, had we not required ϕ and ψ to be unital, we could have chosen (D,ϕ,ψ) so that $\mathrm{rank}(D) = rcp(\pi, \omega\cup\alpha(\omega)\cup\cdots\cup\alpha^{n-1}(\omega), \delta)$, but as $1\in\omega$, any nonunital ϕ and ψ can be rescaled to give unital ones. Hence we find

$$H_\sigma(\gamma, \alpha\circ\gamma, \cdots, \alpha^{n-1}\circ\gamma) \leq \log rcp(\pi, \omega\cup\cdots\cup\alpha^{n-1}(\omega), 4\delta) + n\eta;$$

therefore $h_{\sigma,\alpha}(\gamma) \leq ht(\pi,\alpha,\omega,4\delta)+\eta$. If $\delta\to 0$ then $\eta\to 0$ and we find $h_{\sigma,\alpha}(\gamma)\leq ht(\alpha)$; hence $h_\sigma(\alpha)\leq ht(\alpha)$.

□

Corollary 10. *Let σ_* be the automorphism of the II_1–factor $L(F_\infty)$ induced by an arbitrary permutation σ of the generators of the group F_∞. Then the Connes–Størmer entropy of σ_* is zero.*

Proof. Let τ be the tracial state on $L(F_\infty)$. Combining Example 7(v) with Proposition 9, we find that the CNT–entropy $h_\tau(\sigma_{r,*})$ is zero, where $\sigma_{r,*}$ is the automorphism of $C_r^*(F_\infty)$ arising from the permutation σ of the generators of F_∞ and where τ is the unique tracial state on $C_r^*(F_\infty)$. But $h_\tau(\sigma_{r,*})$ is equal to the CNT–entropy (hence, to the Connes–Størmer entropy) of the corresponding automorphism σ_* of $L(F_\infty)$.

□

The following question is quite natural.

Question 11. Does every pair (D,ψ), where D is a unital exact C*–algebra and where ψ is a conditional expectation from D onto a unital C*–subalgebra, have the ZEFP property?

This seems like an appropriate place to point out that by recent work of Kirchberg [**8**], [**9**], with (D,ψ) as in Question 11, one can always realize $D\subseteq\mathcal{O}_2$; if one could realize $(D,\psi)\subseteq(\mathcal{O}_2,\phi_2)$, with $(\mathcal{O}_2,\phi_2)$ as in Example 7(iii), then by Proposition 6 (D,ψ) would have the ZEFP property.

Support for a positive answer to Question 11 is provided by Størmer's result [**13**] that if D is any unital C*–algebra and ψ is any state on D (with faithful GNS representation), then letting (A,ϕ) be the free product of infinitely many copies of (D,ψ) indexed by a set I, letting σ be a permutation of I without cycles and letting σ_* be the corresponding free permutation of A, the CNT–entropy $h_\phi(\sigma_*)$ of σ_* with respect to the free product state ϕ is zero.

Question 12. Given a reduced free product of C*–algebras $(A,\phi)=(A_1,\phi_1)*(A_2,\phi_2)$, with $\dim(A_1)\geq 2$ and $\dim(A_2)\geq 3$ and where ϕ_1 and ϕ_2 faithful states, is there an automorphism $\alpha\in\mathrm{Aut}(A)$, such that $0<ht(\alpha)<\infty$?

It may be especially interesting to restrict the above question to the case when the states ϕ_1 and ϕ_2 are traces. A first example to consider might be $(A,\tau)=(C^*_r(\mathbf{Z}_2),\tau_{\mathbf{Z}_2})*(C(X),\tau_X)$, where X is the compact Hausdorff space obtained as the product of infinitely many two–element spaces and where τ_X is the state given by the product of uniform measures. Now take $\alpha\in\mathrm{Aut}(A)$ to be $\alpha=\mathrm{id}_{C^*_r(\mathbf{Z}_2)}*\beta$ where β is the Bernoulli shift. Then $ht(\alpha)\geq ht(\beta)=\log 2$. Is $ht(\alpha)$ finite?

Note, however, that it is easy to find a reduced *amalgamated* free product of C*–algebras $(A,\phi)=(A_1,\phi_1)*(A_2,\phi_2)$ with A non–nuclear and $\alpha\in\mathrm{Aut}(A)$ with $0<ht(\alpha)<\infty$. Indeed consider abelian C*–algebras $A_i=C(\mathbf{T})\otimes C(X)$, for some compact Hausdorff space X; let $B=1\otimes C(X)\subseteq A_i$ and let $\phi_i:A_i\to B$ be the slice map obtained from Haar measure on $\mathbf{T}$; then $A=C^*_r(F_2)\otimes C(X)$. Let $\alpha=\mathrm{id}_{C^*(F_2)}\otimes\beta\in\mathrm{Aut}(A)$, where β is an automorphism on $C(X)$ having strictly positive and finite topological entropy. By properties of the Brown–Voiculescu topological entropy [**2**], we have $ht(\beta)\leq ht(\alpha)\leq ht(\mathrm{id}_{C^*_r(F_2)})+ht(\beta)=ht(\beta)$.

References

1. E. Blanchard, K.J. Dykema, *Embeddings of reduced free products of operator algebras*, preprint (1999).
2. N. Brown, *Topological entropy in exact C^*–algebras*, Math. Ann. (to appear).
3. N. Brown, M. Choda, *Approximation entropies in crossed products with an application to free shifts*, preprint (1999).
4. A. Connes, H. Narnhofer, W. Thirring, *Dynamical approximation entropies of C^*–algebras and von Neumann algebras*, Commun. Math. Phys. **112** (1987), 691-719.
5. A. Connes, E. Størmer, *Entropy for automorphisms of II_1 von Neumann algebras*, Acta Math. **134** (1975), 289-306.
6. J. Cuntz, *Simple C^*–algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173-185.
7. K.J. Dykema, *Exactness of reduced amalgamated free products of C^*–algebras*, preprint (1999).
8. E. Kirchberg, *Exact C^*–algebras, tensor peoructs and the classification of purely infinite algebras*, Proceedings of the International Congress of Mathematicians (Zürich, 1994), Birkhäuser Verlag, 1995, pp. 943-954.
9. E. Kirchberg, N.C. Phillips, *Embeddings of exact C^*–algebras and continuous fields in the Cuntz algebra $\mathcal{O}_2$*, preprint (1997).
10. J.-L. Sauvageot, J.-P. Thouvenot, *Une nouvelle définition de l'entropie dynamique des systèmes non commutatifs*, Commun. Math. Phys. **145** (1992), 411-423.
11. E. Størmer, *Entropy of some automorphisms of the II_1–factor of the free group in infinite number of generators*, Invent. Math. **110** (1992), 63-73.
12. ______, *Entropy in operator algebras*, Recent Advances in Operator Algebras, Orléans 1992, Astérisque, vol. 232, Soc. Math. France, 1995, pp. 211-230.
13. ______, *States and shifts on infinite free products of C^*–algebras*, Fields Inst. Commun. **12** (1997), 281-291.
14. D. Voiculescu, *Symmetries of some reduced free product C^*–algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, vol. 1132, Springer–Verlag, 1985, pp. 556–588.
15. ______, *Dynamical approximation and topological entropies in operator algebras*, Commun. Math. Phys. **170** (1995), 249-281.
16. D. Voiculescu, K.J. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series vol. 1, American Mathematical Society, 1992.

Dept. of Mathematics, Texas A&M University, College Station TX 77843-3368, USA
E-mail address: `Ken.Dykema@math.tamu.edu`, *Internet URL:* `http://www.math.tamu.edu/~Ken.Dykema/`